Calculus 2 - Test 3 Review Key

Dr. Graham-Squire, Spring 2013

1. Consider the region W bounded by $y = \frac{1}{x}$, $y = 0$, $x = 1$ and $x = 3$. Find the volume of the solid obtained by rotating W about (a) the line $y = -3$ and (b) the y-axis.

   Ans: (a) Using washers, get $\displaystyle\int_1^3 \pi[(3 + \frac{1}{x})^2 - 3^2]\,dx = \pi(6\ln 3 + (2/3))$.

   (b) Using shells, get $\displaystyle\int_1^3 2\pi x(\frac{1}{x})\,dx = 4\pi$.

2. Calculate the arc length of the curve $y = 4(x-3)^{3/2}$ for $3 \leq x \leq \frac{37}{12}$.

   Ans: Can take the integral by hand. Final answer is 7/54.

3. A tank has the shape of an inverted circular cone with height 10 meters and base radius 4 meters. It is filled with water to a height of 8 meters. Note: The density of water is 1000 kg/cubic meter, and gravity is 9.8 m/sec^2.

   (a) Find the work required to empty the tank by pumping all of the water to the top of the tank.

   Ans: $1568\pi \displaystyle\int_0^8 x^2(10 - x)dx = 107041.3\pi$

   (b) Find the work required to pump the water to a point that is 6 meters above the top of the tank.

   Ans: $1568\pi \displaystyle\int_0^8 x^2(16 - x)dx = 2760533.3\pi$

4. We have a cable that weighs 3 lbs/ft attached to a bucket filled with coal that weighs 700 lbs. The bucket is initially at the bottom of a 600 ft mine shaft. Answer each of the following.

   (a) Determine the amount of work required to lift the bucket to the midpoint of the shaft.

   Ans: Work for top half of rope + Work for bottom half of rope + work for bucket of coal = $\displaystyle\int_0^{300} 3x\,dx + 300(3)(300) + 700(300) = 615000$

   (b) Determine the amount of work required to lift the bucket from the midpoint of the shaft to the top of the shaft

   Ans: $\displaystyle\int_0^{300} 3x\,dx + 700(300) = 345000$

   (c) Determine the amount of work required to lift the bucket all the way up the shaft.

   Ans: $\displaystyle\int_0^{600} 3x\,dx + 700(600) = 960000.$

5. Determine whether the sequence is convergent or divergent. If convergent, find the limit.

(a) $a_n = \dfrac{n^3 + n^2 \cos n}{n^3}$

**ANS: Converges to 1. Split it into two fractions and use comparison.

(b) $b_n = \dfrac{\sqrt[3]{n}}{\ln n}$. **ANS: Diverges. Use L'Hospital's rule.

6. Determine if the series is convergent or divergent. If it is convergent, find the limit. Make sure you state which convergence/divergence test you use (or if no test, then explain your reasoning).

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{5^{n+2}}$. **ANS: Geometric series. Converges to 3/50.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n(n+2)}$ Note: You may need to use a partial fraction decomposition.

**ANS: Telescoping series. Converges to 3/2.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+8}{(10n+1)^2}$

**ANS: The limit of the terms approaches $3/100 \neq 0$, so the series diverges by the test for divergence.

7. Test the series for convergence or divergence. If it converges, state whether it is absolutely convergent or not.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{(-5)^n - 7}{4^n}$

**ANS: Diverges. Split it into two fractions, one is geometric with $|r| > 1$ so it diverges (the other converges, but it does not matter).

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt[3]{n^4}+10}{n^2}$

**ANS: Converges by the alternating series test, but is not absolutely convergent (to see it is not absolutely convergent you can do a limit comparison with $1/n^{(2/3)}$, which is a divergent p-series).

(c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^4}{3^n}$

**ANS: Use the ratio test, will be absolutely convergent.

8. (a) Find the sum of $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$ correct to four decimal places.

**ANS: Use the alternating series remainder test to see that you only need to add up the first ten terms, that is $1 - \frac{1}{16} + \frac{1}{81} - \cdots - \frac{1}{10{,}000}$.

(b) How would you find the approximation if it was a 1 in the numerator instead of $(-1)^{n+1}$ (That is, how would you figure out how many terms you need to add up)?

**ANS: Need to use the integral remainder test and find for what value of k you have $\displaystyle\int_k^{\infty} \frac{1}{x^4}\,dx$ less than 0.0001, which is $k > \sqrt[3]{1/0.0003} = 14.93$, so $k = 15$.